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Explicit solutions of the fifth-order KdV type nonlinear evolution equation using the system technique

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ABSTRACT

We consider the generalized fifth-order KdV type nonlinear evolution equation with variable coefficients. The system technique has been applied rigorously in order to find new exact solutions of the considered equations. The closed-form solutions of the fifth-order KdV type nonlinear evolution equation are expressed by the proposed ansätz in terms of exponential functions. We believe that the system technique is effective and stable of finding the exact solutions of nonlinear evolution equations. Further, we describe the behaviors of the obtained solutions under certain constraints and variable coefficients.

Introduction

Nonlinear partial differential equations play an important role in understanding physical phenomena in applied sciences such as physics, engineering, chemistry and biology. A great deal of attention has been paid towards the exact solutions of nonlinear partial differential equations by many researchers. Thus, it is important to find new mathematical algorithms for determining the exact solutions of nonlinear partial differential equations. It is noted that powerful methods such as the extended Jacobi elliptic function expansion method [1], Kronecker product technique [2], tanh-expansion method [3–5], (G'/G)-expansion method [6,7], Kudryashov method [8–14], and so on, have been proposed to handle nonlinear evolution equations. On the other hand, finding exact solutions are very important for understanding the internal mechanism of physical phenomena, and the closed-form solutions of nonlinear partial differential equations can assist numerical solvers at comparing the correctness of their results and help the stability analysis.

In the present study, we employ the system technique with a different mathematical tool to obtain new exact solutions for nonlinear partial differential equations that contain exponential functions based on some suitable choices of parameters. This method is very effective and powerful for finding the exact solutions of nonlinear partial differential equations and shows the validity and potentiality of the system technique for obtaining exact solutions of nonlinear partial differential equations, in particular, the nonlinear higher-dimensional physical models and coupled partial differential equations arising in mathematical physics, science and engineering. The main work of this paper is to obtain more exact solutions of nonlinear partial differential equations such as the fifth-order KdV type nonlinear partial differential equations, using the system technique and to describe the behaviors of the obtained exact variable coefficients [15]. We will be applying this technique in a variety of fields such as the stochastic wick-type nonlinear differential equations with white functionals and we will improve the system technique to find the exact solutions of nonlinear partial differential equations.

This paper is organized as follows. In Section "The algorithm for solving nonlinear partial differential equation through the system technique", we describe the algorithm for the system technique that finds exact solutions of general nonlinear partial differential equations. In Section "The fifth-order KdV type nonlinear evolution equations", we represent new exact solutions of the fifth-order KdV type nonlinear evolution equations with variable coefficients and some special cases of the fifth-order KdV type nonlinear evolution equations and provide graphs of these solutions with analysis. Finally, some conclusions are given.

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The algorithm for solving nonlinear partial differential equation through the system technique

In this section, we present an outline of the system technique for obtaining exact solutions of nonlinear partial differential equations. Suppose that the nonlinear partial differential equation, say in the variables $x_1, x_2, \ldots, x_n$ and t, is given by

$$\mathcal{P}(u, u_t, u_{x_1}, \ldots, u_{x_n}, u_{tt}, u_{x_1x_1}, u_{x_1x_2}, \ldots) = 0, \tag{1}$$

where $u = u(x_1, x_2, \ldots, x_n, t)$ is an unknown function, $\mathcal{P}$ is a polynomial in u and its various partial derivatives, in which the highest order derivatives and the nonlinear terms are involved.

Using the traveling wave variable $z = \int_0^t k_1(\tau)d\tau x_1 + \int_0^t k_2(\tau)d\tau x_2 + \cdots + \int_0^t k_n(\tau)d\tau x_n + \int_0^t \omega(\tau)d\tau$, we can express this as an unknown function

$$u(x_1, x_2, \ldots, x_n, t) = \mathrm{u}(z), \tag{2}$$

and we can convert Eq. (1) into an ordinary differential equation for $\mathrm{u} = \mathrm{u}(z)$ as follows:

$$\mathcal{Q}(\mathrm{u}, \mathrm{u}_z, \mathrm{u}_{zz}, \mathrm{u}_{zzz}, \ldots) = 0. \tag{3}$$

where $\mathrm{u}_z = \frac{d\mathrm{u}}{dz}, \mathrm{u}_{zz} = \frac{d^2\mathrm{u}}{dz^2}, \mathrm{u}_{zzz} = \frac{d^3\mathrm{u}}{dz^3}$, and so on.

To find solution u explicitly, by the homogeneous balancing property in Eq. (3), we have the m-order pole of the solution and so we suppose that the solution of Eq. (3) can be expressed as a polynomial in $\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)$ as follows:

$$\mathrm{u}(z) = A_m\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)^m + A_{m-1}\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)^{m-1} + \cdots + A_1\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right) + A_0, \tag{4}$$

where $\mathrm{F}(z)$ and $\mathrm{G}(z)$ satisfy the following system:

$$\begin{cases} \mathrm{F}'(z) = p\mathrm{F}(z), \\ \mathrm{G}'(z) = p\mathrm{F}(z) + q\mathrm{G}(z), \end{cases} \tag{5}$$

and $A_m, A_{m-1}, \ldots, A_0, k_1, \ldots, k_n$ and ω are constants to be determined later with $A_m \neq 0$. The unwritten part in (4) is also a polynomial in $(\mathrm{F}(z)/\mathrm{G}(z))$, the degree of which is generally equal to or less than $m-1$, and the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear term appearing in Eq. (3).

The system Eq. (5) admits the following ansätz of the solution

$$\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right) = \frac{p-q}{p - q\exp\{-(p-q)z\}}, \tag{6}$$

where p and q are nonzero constants with $p \neq q$, as shown in Remark 2.1.

Then, by substituting Eq. (4) into Eq. (3) and collecting all terms with the same order of $(\mathrm{F}(z)/\mathrm{G}(z))$ together, the left-hand side of Eq. (3) is converted into another polynomial in $(\mathrm{F}(z)/\mathrm{G}(z))$ using derivatives, as seen in Remark 2.2. Equating each coefficient of this polynomial to zero, we yield a set of algebraic equations for the coefficients $A_m, A_{m-1}, \ldots, A_0, k_1, \ldots, k_n$ and ω.

Last, assuming that the coefficients $A_m, A_{m-1}, \ldots, A_0, k_1, \ldots, k_n$ and ω can be obtained by solving the algebraic equations in the polynomial, then by substituting $A_m, A_{m-1}, \ldots, A_0, k_1, \ldots, k_n$ and ω into Eq. (4), we have new exact solutions of the nonlinear evolution Eq. (1) via the wave transformation Eq. (2).

Remark 2.1. The integrability conditions of the system, given by Eq. (5), have been extensively discussed in the literature and the relations among the coefficients of the system involve two constraints as follows: $p \neq q$ and $p = q$. First, we consider the solution of Eq. (5) for $p \neq q$. Assuming $\mathrm{F}(0) = 1$ and $\mathrm{G}(0) = 1$. The first differential equation of Eq. (5), the solution $\mathrm{F}(z)$ is given by

$$\mathrm{F}(z) = e^{pz}. \tag{7}$$

Substituting Eq. (7) into the second equation of Eq. (5), we obtain a nonhomogeneous first order linear differential equation

$$\mathrm{G}'(z) - q\mathrm{G}(z) = pe^{pz}. \tag{8}$$

Then, we can find the solution of Eq. (8) as follows:

$$\mathrm{G}(z) = e^{qz}\left[\frac{p}{p-q}e^{-(p-q)z} - \frac{q}{p-q}\right]. \tag{9}$$

Combining Eqs. (7) and (9), we obtain the following proposed function:

$$\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right) = \frac{p-q}{p - q\exp\{-(p-q)z\}}, \tag{10}$$

where p and q are arbitrary nonzero constants with $p \neq q$.

On the other hand, we have another proposed function;

$$\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right) = \frac{1}{z+1}. \tag{11}$$

Remark 2.2. The following derivatives are useful for equating the expressions at the same degrees of (F/G) to zero in Eq. (3); $(\mathrm{F}/\mathrm{G})' = (p-q)(\mathrm{F}/\mathrm{G}) - p(\mathrm{F}/\mathrm{G})^2$, $(\mathrm{F}/\mathrm{G})'' = (p-q)^2(\mathrm{F}/\mathrm{G}) - 3p(p-q)(\mathrm{F}/\mathrm{G})^2 + 2p^2(\mathrm{F}/\mathrm{G})^3$, $(\mathrm{F}/\mathrm{G})''' = (p-q)^3(\mathrm{F}/\mathrm{G}) - 7p(p-q)^2(\mathrm{F}/\mathrm{G})^2 + 12p^2(p-q)(\mathrm{F}/\mathrm{G})^3 - 6p^3(\mathrm{F}/\mathrm{G})^4$, $(\mathrm{F}/\mathrm{G})^{(4)} = (p-q)^4(\mathrm{F}/\mathrm{G}) - 15p(p-q)^3(\mathrm{F}/\mathrm{G})^2 + 50p^2(p-q)^2(\mathrm{F}/\mathrm{G})^3 - 60p^3(p-q)(\mathrm{F}/\mathrm{G})^4 + 24p^4(\mathrm{F}/\mathrm{G})^5$, $(\mathrm{F}/\mathrm{G})^{(5)} = (p-q)^5(\mathrm{F}/\mathrm{G}) - 31p(p-q)^4(\mathrm{F}/\mathrm{G})^2 + 180p^2(p-q)^3(\mathrm{F}/\mathrm{G})^3 - 390p^3(p-q)^2(\mathrm{F}/\mathrm{G})^4 + 360p^4(p-q)(\mathrm{F}/\mathrm{G})^5 + 120p^5(\mathrm{F}/\mathrm{G})^6$, and so on.

The fifth-order KdV type nonlinear evolution equations

In this section, we will employ the system technique to obtain new explicit solutions of the generalized fifth-order KdV type nonlinear evolution equation with variable coefficients, given by

$$u_t + \alpha(t)u^2u_x + \beta(t)u_xu_{xx} + \gamma(t)uu_{xxx} + u_{xxxxx} = 0, \tag{12}$$

where u is a function of the spatial variable x and the time variable t, and the variable coefficients $\alpha(t), \beta(t)$ and $\gamma(t)$ are arbitrary functions. Eq. (12) describes phenomena in quantum mechanics and nonlinear optics. This equation is the most popular soliton equation and often exists in practical problems such as fluid physics and quantum field theory [16–20].

Using the wave transformation $u(x,t) = \mathrm{u}(z), z = \int_0^t k(\tau)d\tau x + \int_0^t \omega(\tau)d\tau$, Eq. (12) is converted into the following

$$\omega(t)\mathrm{u}_z + \alpha(t)k(t)\mathrm{u}^2\mathrm{u}_z + \beta(t)k^3(t)\mathrm{u}_z\mathrm{u}_{zz} + \gamma(t)k^3(t)\mathrm{u}\mathrm{u}_{zzz} + k^5(t)\mathrm{u}_{zzzzz} = 0, \tag{13}$$

where $\mathrm{u}_z = \frac{d\mathrm{u}}{dz}, \mathrm{u}_{zz} = \frac{d^2\mathrm{u}}{dz^2}, \mathrm{u}_{zzz} = \frac{d^3\mathrm{u}}{dz^3}$ and $\mathrm{u}_{zzzzz} = \frac{d^5\mathrm{u}}{dz^5}$.

Let us employ the system technique Eq. (5) for finding the exact solution of Eq. (13). Balancing the highest order nonlinear term $\mathrm{u}^2\mathrm{u}_z$ and the highest order linear term u_{zzzzz} in Eq. (13), we obtain $3m + 1 = m + 5$ which gives $m = 2$. Then Eq. (13) has a second-order pole solution by the homogeneous balancing principle. We suppose the exact solution of Eq. (13) can be expressed in the form of

$$\mathrm{u}(z) = A_0(t) + A_1(t)\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right) + A_2(t)\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)^2, \tag{14}$$

where $A_0(t), A_1(t), A_2(t), \beta(t), \gamma(t), k(t)$ and $\omega(t)$ are arbitrary functions to be determined later.

By taking the exact solution Eq. (14) and the system Eq. (5), we can obtain the derivatives u_z, u_{zz}, u_{zzz} and u_{zzzzz} expressed in terms of $(F(z)/G(z))$, as in Remark 2.2. By substituting u_z, u_{zz}, u_{zzz} and u_{zzzzz} into the left-hand side of Eq. (13), the left-hand side of Eq. (13) is converted into a polynomial in (F/G). Equating each coefficient of this polynomial to zero and solving this system with the help of Maple, we obtain the following four sets of nontrivial solutions:

$$\begin{cases} k(t) = \pm\int_0^t \frac{1}{p}\left\{-\frac{A_2^2(\tau)\alpha(\tau)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} d\tau, \\ \omega(t) = \pm\int_0^t \frac{K_\omega A_2^2(\tau)\alpha(\tau)}{36p^5(p-q)^2}\left\{-\frac{A_2^2(\tau)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} d\tau, \\ \beta(t) = \frac{A_2(t)\alpha(t)(p^2+q^2)}{6(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}, \\ \gamma(t) = \frac{2A_2(t)\alpha(t)pq}{3(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}, \\ A_0(t) = -\frac{A_2(t)(p^2-5q^2)}{p^2}, \quad A_1(t) = \frac{2A_2(t)q}{p}, \quad A_2(t) = A_2(t), \end{cases} \tag{15}$$

$$\begin{cases} k(t) = \pm\int_0^t \frac{1}{p}\left\{-\frac{A_2^2(\tau)\alpha(\tau)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau, \\ \omega(t) = \pm\int_0^t \frac{K_\omega A_2^2(\tau)\alpha(\tau)}{36p^5(p-q)^2}\left\{-\frac{A_2^2(\tau)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau, \\ \beta(t) = -\frac{A_2(t)\alpha(t)(p^2+q^2)}{6(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}, \\ \gamma(t) = -\frac{2A_2(t)\alpha(t)pq}{3(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}, \\ A_0(t) = -\frac{A_2(t)(p^2-5q^2)}{p^2}, \quad A_1(t) = \frac{2A_2(t)q}{p}, \quad A_2(t) = A_2(t), \end{cases} \tag{16}$$

$$\begin{cases} k(t) = k(t), \omega(t) = \int_0^t -\frac{30k^5(\tau)}{2p-3q}\left\{\frac{(A_{22}+5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}+5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\} d\tau, \\ \beta(t) = \pm\frac{(60(A_{22}+5\sqrt{K_1})(p-4q)-120A_{21}(p-12q))\sqrt{15}\alpha(t)}{300p(\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}}, \\ \gamma(t) = \mp\frac{(60(A_{22}+5\sqrt{K_1})(2p-3q)+360A_{21}(p+3q))\sqrt{15}\alpha(t)}{450p(\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}}, \\ A_0(t) = \mp pk^2(t)(15\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}} \\ \quad \times\left\{\frac{60(A_{22}+5\sqrt{K_1})(p^2-4pq+5q^2)+120(45q^2-p^2-8pq)A_{21}}{\alpha(t)A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\}, \\ A_1(t) = 0, \quad A_2(t) = \pm\frac{2p^2k^2(t)(15\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}}{\alpha(t)A_{21}} \end{cases} \tag{17}$$

and

$$\begin{cases} k(t) = k(t), \omega(t) = \int_0^t -\frac{30k^5(\tau)}{2p-3q}\left\{\frac{(A_{22}-5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}-5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\} d\tau, \\ \beta(t) = \pm\frac{\alpha(t)(60(A_{22}-5\sqrt{K_1})(p-4q)-120A_{21}(p-12q))}{20p(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}}, \\ \gamma(t) = \mp\frac{\alpha(t)(60(A_{22}-5\sqrt{K_1})(2p-3q)+360A_{21}(p+3q))}{30p(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}}, \\ A_0(t) = \mp pk^2(t)(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}} \\ \quad \times\left\{\frac{60(A_{22}-5\sqrt{K_1})(p^2-4pq+5q^2)+120(45q^2-p^2-8pq)A_{21}}{\alpha(t)A_{21}(60(A_{22}-5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\}, \\ A_1(t) = 0, \quad A_2(t) = \pm\frac{2p^2k^2(t)(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}}{\alpha(t)A_{21}}. \end{cases} \tag{18}$$

In the above nontrivial solutions we have: $K_\omega = (p^2+q^2)(p^4-2q^3p-10p^2q^2-12p^3q+q^4), A_{21} = 2p^4+60q^4 -103q^3p-19qp^3+66p^2q^2, A_{22} = -p^4+20p^3q-187p^2q^2+342pq^3$, $K_1 = p^8-4p^7q-6p^6q^2-100p^5q^3+1561p^4q^4-6024p^3q^5+9792p^2q^6+5184q^8-8640pq^7, K_{\omega1} = 180p^{11}-1260p^{10}q+2340p^9q^2+15060p^8q^3-140820p^7q^4+646140p^6q^5-905940p^5q^6-4211460p^4q^7+18979920p^3q^8-29665440p^2q^9+20528640pq^{10}-5598720q^{11}, K_{\omega2} = -360p^{11}-1800p^{10}q+15480p^9q^2+47160p^8q^3-627480p^7q^4+1014120p^6q^5+3519720p^5q^6-15289560p^4q^7+15202080p^3q^8+23561280p^2q^9-61585920pq^{10}+33592320q^{11}$, $K_{\omega3} = 77760p^2q^4-240p^5q+15520q^6-336960pq^5+300p^6-1014p^4q^2+35640p^3q^3, K_{\omega4} = -933120q^6+17640p^4q^2-1800p^6-38880q^4p^2+4680p^5q-119880p^3q^3+311040q^5p$.

With the wave transformation $z = \int_0^t k(\tau)d\tau x + \int_0^t \omega(\tau)d\tau$ and Eq. (14), we can obtain the exact solutions of Eq. (12), which can be written as follows.

For the nontrivial solution set Eq. (15), with some relations as

$$\beta(t) = \frac{A_2(t)\alpha(t)(p^2+q^2)}{6(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}},$$

$$\gamma(t) = \frac{2A_2(t)\alpha(t)pq}{3(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}, \tag{19}$$

the exact solution of Eq. (12) can be written by

$$u_1(x,t) = -\frac{A_2(t)(p^2-5q^2)}{p^2} + \frac{2A_2(t)q}{p}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right) + A_2(t)\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \tag{20}$$

where $z = \pm\int_0^t \frac{1}{p}\left\{-\frac{A_2^2(\tau)\alpha(\tau)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau x \pm \int_0^t \frac{K_\omega A_2^2(\tau)\alpha(\tau)}{36p^5(p-q)^2}\left\{-\frac{A_2^2(\tau)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau$ with some relations

The second explicit solution of Eq. (12) for the solution set Eq. (16) is a complex solution in the form

$$u_2(x,t) = -\frac{A_2(t)(p^2-5q^2)}{p^2} + \frac{2A_2(t)q}{p}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right) + A_2(t)\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \tag{21}$$

where $z = \pm\int_0^t \frac{1}{p}\left\{-\frac{A_2^2(\tau)\alpha(\tau)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau x \pm \int_0^t \frac{K_\omega A_2^2(\tau)\alpha(\tau)}{36p^5(p-q)^2}\left\{-\frac{A_2^2(\tau)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{\frac{1}{4}} i d\tau$ with some relations

$$\beta(t) = -\frac{A_2(t)\alpha(t)(p^2+q^2)}{6(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}},$$

$$\gamma(t) = -\frac{2A_2(t)\alpha(t)pq}{3(p-q)^2}\left\{-\frac{A_2^2(t)\alpha(t)(p^2+3pq+q^2)}{180(p-q)^2}\right\}^{-\frac{1}{2}}. \tag{22}$$

The third explicit solution of Eq. (12) for the solution set Eq. (17) is expressed in the form

$$u_3(x,t) = \mp\left(15\alpha(t)A_{21}(A_{22}+5\sqrt{K_1})\right)^{\frac{1}{2}} pk^2(t) \times\left\{\frac{60(A_{22}+5\sqrt{K_1})(p^2-4pq+5q^2)+120(45q^2-p^2-8pq)A_{21}}{\alpha(t)A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\} \pm\frac{2\sqrt{5}(\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{\alpha(t)A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \tag{23}$$

where $z = \int_0^t k(\tau)d\tau x - \int_0^t \frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}+5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}+5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\}d\tau$ and

$$\beta(t) = \pm\frac{(60(A_{22}+5\sqrt{K_1})(p-4q)-120A_{21}(p-12q))\sqrt{15}\alpha(t)}{300p(\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}},$$

$$\gamma(t) = \mp\frac{(60(A_{22}+5\sqrt{K_1})(2p-3q)+360A_{21}(p+3q))\sqrt{15}\alpha(t)}{450p(\alpha(t)A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}}. \tag{24}$$

The last exact solution of Eq. (12) for the nontrivial solution set Eq. (18) is given in the form

$$u_4(x,t)=\mp\left(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1})\right)^{\frac{1}{2}}pk^2(t)\times\left\{\frac{60(A_{22}-5\sqrt{K_1})(p^2-4pq+5q^2)+120(45q^2-p^2-8pq)A_{21}}{\alpha(t)A_{21}(60(A_{22}-5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\pm\frac{2\sqrt{5}(\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{\alpha(t)A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2,\tag{25}$$

where $z=\int_0^t k(\tau)d\tau x-\int_0^t\frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}-5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}-5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\}d\tau$ and

$$\beta(t)=\pm\frac{\alpha(t)(60(A_{22}-5\sqrt{K_1})(p-4q)-120A_{21}(p-12q))}{20p(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}},\quad \gamma(t)=\mp\frac{\alpha(t)(60(A_{22}-5\sqrt{K_1})(2p-3q)+360A_{21}(p+3q))}{30p(15\alpha(t)A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}}.\tag{26}$$

Example 3.1. We have the following explicit solution of the exact solution Eq. (20) with the coefficient $A_2(t)$ under the conditions $p=-0.14, q=-0.52, \alpha(t)=t^2$, given by

$$u_1(x,t)=67.9796A_2(t)+\frac{2.82286A_2(t)}{-0.14+0.52\exp\{-0.38(3.98891t^{\frac{7}{2}}A_2^{\frac{5}{2}}(t)-1.78083t^{\frac{3}{2}}A_2^{\frac{1}{2}}(t)x)\}}+\frac{0.1444A_2(t)}{(-0.14+0.52\exp\{-0.38(3.98891t^{\frac{7}{2}}A_2^{\frac{5}{2}}(t)-1.78083t^{\frac{3}{2}}A_2^{\frac{1}{2}}(t)x)\})^2}.\tag{27}$$

In Fig. 1, the behaviors of the explicit solution Eq. (27) are as follows: part (a) is a solitary wave solution with two peaks for $A_2(t)=0.1$, part (b) shows a traveling wave solution with kinks for $A_2(t)=0.001$, and part (c) is a soliton-like solution with two peaks for $A_2(t)=0.0001$.

Remark 3.2. We know that the spacial coefficient $k(t)$, the traveling speed coefficient $\omega(t)$, variable coefficients $\beta(t),\gamma(t)$ of Eq. (12) and the coefficients $A_0(t),A_1(t),A_2(t)$ of the solution Eq. (14) depend on the variable coefficient $\alpha(t)$ and the parameters p,q of Eq. (5) from the nontrivial solution sets Eqs. (15)–(18).

Remark 3.3. From Eq. (19) of the exact solution Eqs. (20) and (22) of the exact solution Eq. (21), we can obtain the relation between $\beta(t)$ and $\gamma(t)$, which is $\beta(t)=\frac{p^2+q^2}{4pq}\gamma(t)$, depending on the parameters p,q of Eq. (5). Also, based on Eq. (24) of the exact solution Eqs. (23) and (26) of the exact solution Eq. (25), there are two relations, $\beta(t)=-\frac{3(A_{22}+5\sqrt{K_1})(p-4q)-6A_{21}(p-12q)}{2(A_{22}+5\sqrt{K_1})(2p-3q)+12A_{21}(p+3q)}\gamma(t)$ and $\beta(t)=-\frac{3(A_{22}-5\sqrt{K_1})(p-4q)-6A_{21}(p-12q)}{2(A_{22}-5\sqrt{K_1})(2p-3q)+12A_{21}(p+3q)}\gamma(t)$, depending on the parameters p,q of the system Eq. (5), respectively.

Example 3.4. The Caudrey-Dodd-Gibbon(CDG) equation is given by [21,22]

$$u_t+180u^2u_x+30u_xu_{xx}+30uu_{xxx}+u_{xxxxx}=0.\tag{28}$$

It is obvious that the constant variable coefficients correspond to $\alpha(t)=180, \beta(t)=\gamma(t)=30$ in Eq. (12) for the CDG equation. This is associated with scattering problems. Based on these values of $\alpha(t)=180, \beta(t)=\gamma(t)=30$, there are some special solutions of Eq. (28) as follow. First, from the relations Eq. (19) of variable coefficients in the solution Eq. (20), we can obtain a new relation between p and q as $q=2p\pm\sqrt{3}|p|$ in Remark 3.3 and then we can rewrite a new exact solution in the form

$$u_{11}(x,t)=A_2(t)\frac{34p^2\pm20\sqrt{3}p|p|}{p^2}+A_2(t)\frac{2(2p\pm\sqrt{3}|p|)}{p}\left(\frac{\mathsf{F}(z)}{\mathsf{G}(z)}\right)+A_2(t)\left(\frac{\mathsf{F}(z)}{\mathsf{G}(z)}\right)^2,\tag{29}$$

where $z=\pm\left(\frac{-14p^2\mp7\sqrt{3}p|p|}{(p\pm\sqrt{3}|p|)^2}\right)^{\frac{1}{4}}\left\{\int_0^t\frac{1}{p}\left(A_2^2(\tau)\right)^{\frac{1}{4}}d\tau x+\int_0^t\frac{5K_\omega A_2^2(\tau)}{p^5(p\pm\sqrt{3}|p|)^2}\left(A_2^2(\tau)\right)^{\frac{1}{4}}d\tau\right\}$ and $\left(\frac{\mathsf{F}(z)}{\mathsf{G}(z)}\right)=\frac{-p\mp\sqrt{3}|p|}{p-(2p\pm\sqrt{3}|p|)\exp\{(p\pm\sqrt{3}|p|)z\}}$ and $A_2(t)$ is an arbitrary function.

Otherwise, in constraints Eqs. (24) and (26) of the solutions Eqs. (23) and (25), respectively, we can derive the same conditions of p and q as $p\neq4q$ and $2p\neq3q$ and then we can rewrite the exact solutions in the forms

$$u_{31}(x,t)=\mp30\sqrt{3}\left(A_{21}(A_{22}+5\sqrt{K_1})\right)^{\frac{1}{2}}pk^2(t)\times\left\{\frac{(A_{22}+5\sqrt{K_1})(p^2-4pq+5q^2)+2(45q^2-p^2-8pq)A_{21}}{3A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\pm\frac{(A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{3A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2,\tag{30}$$

where $z=\int_0^t k(\tau)d\tau x-\int_0^t\frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}+5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}+5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\}d\tau$ and $k(t)$ is an arbitrary function,

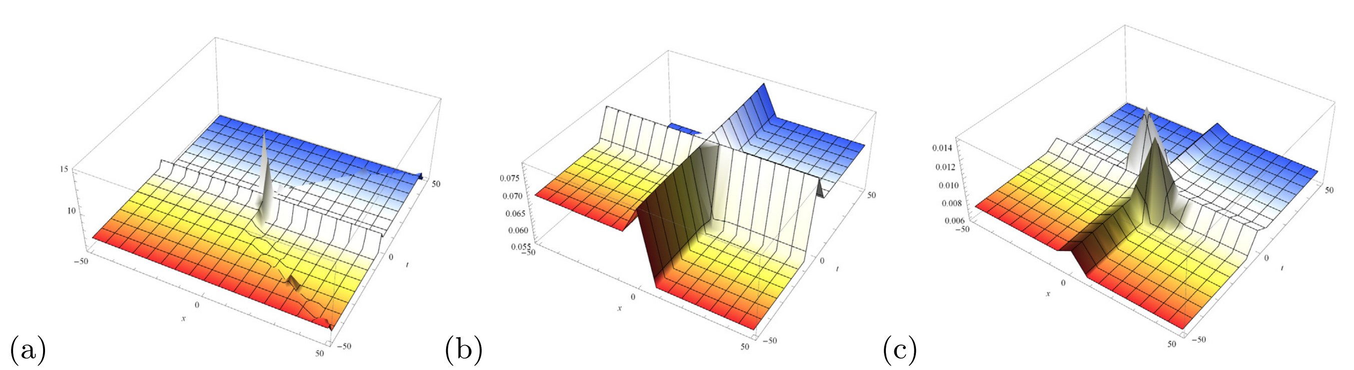

Fig. 1. Profiles of the exact solution Eq. (27): (a) a solitary solution with two peaks for $A_2(t)=0.1$, (b) a traveling wave solution with kinks for $A_2(t)=0.001$, (c) a solitons-like solution for $A_2(t)=0.0001$, all under $p=-0.14, q=-0.52, \alpha(t)=t^2$.

$$u_{41}(x,t)=\mp30\sqrt{3}\Big(A_{21}(A_{22}-5\sqrt{K_1})\Big)^{\frac{1}{2}}pk^2(t)\times\left\{\frac{(A_{22}-5\sqrt{K_1})(p^2-4pq+5q^2)+2(45q^2-p^2-8pq)A_{21}}{3A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\}\pm\frac{(A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{3A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \quad (31)$$

where $z=\int_0^t k(\tau)d\tau x-\int_0^t\frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}-5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}-5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\}d\tau$ and $k(t)$ is an arbitrary function.

We discuss the behaviors of the obtained solution Eq. (20) in Fig. 2 through Fig. 5 via the computer simulations as follows. Fig. 2 shows the behaviors of the solitary wave solution Eq. (20) for various constant values of the coefficient $A_2(t)$: part (a) shows the motion with one peak for $A_2(t)=0.1$, part (b) is the traveling wave solution with kinks for $A_2(t)=0.01$ and part (c) represents a solitons-like solution with two peaks for $A_2(t)=0.001$, under the values of $p=-1, q=-2+\sqrt{3}$. Fig. 3 consists of the behaviors of the exact solution Eq. (20): (a) soliton solution for $A_2(t)=0.1$, (b) one peak solution for $A_2(t)=0.01$ and (c) solution with kinks for $A_2(t)=0.001$, under $p=-1, q=-2-\sqrt{3}$. Assuming the values of $p=-0.1, q=-0.1(2+\sqrt{3})$, the exact solution Eq. (20) has three different periodic behaviors shown in Fig. 4, part (a) is a periodic traveling wave solution with peaks when $A_2(t)=\sin^2(0.1t)$, part (b) and (c) show periodic traveling wave motions for $A_2(t)=\cos^2(0.1t)$ and $A_2(t)=1$, respectively.

Example 3.5. The Lax equation is a special case of the fifth-order KdV equation as follows:

$$u_t+30u^2u_x+20u_xu_{xx}+10uu_{xxx}+u_{xxxxx}=0, \quad (32)$$

which is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory [23–25].

It is obvious that constant variable coefficients correspond to $\alpha(t)=30, \beta(t)=20, \gamma(t)=10$ in Eq. (12) for the Lax equation. Based on these values of $\alpha(t)=30, \beta(t)=20, \gamma(t)=10$, there are some special solutions of Eq. (32) as follow. From the relations Eq. (19) of variable coefficients in solution Eq. (20), we can obtain a new relation between p and q as $q=4p\pm\sqrt{15}|p|$ (Remark 3.3) and then we can rewrite a new exact solution in the form of

$$u_{12}(x,t)=A_2(t)\frac{154p^2\pm40\sqrt{15}p|p|}{p^2}+A_2(t)\times\frac{2(4p\pm\sqrt{15}|p|)}{p}\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)+A_2(t)\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)^2, \quad (33)$$

where $z=\pm\left(\frac{-44p^2\mp11\sqrt{15}p|p|}{6(3p\pm\sqrt{15}|p|)^2}\right)^{\frac{1}{4}}\left\{\int_0^t\frac{1}{p}\left(A_2^2(\tau)\right)^{\frac{1}{4}}d\tau x+\int_0^t\frac{5K_\omega A_2^2(\tau)}{p^5(3p\pm\sqrt{15}|p|)^2}\left(A_2^2(\tau)\right)^{\frac{1}{4}}d\tau\right\}$ and $\left(\frac{\mathrm{F}(z)}{\mathrm{G}(z)}\right)=\frac{-3p\mp\sqrt{15}|p|}{p-(4p\pm\sqrt{15}|p|)\exp\{(3p\mp\sqrt{15}|p|)z\}}$ and $A_2(t)$ is an arbitrary function.

Otherwise, in constraints Eqs. (24) and (26) of solutions Eqs. (23) and (25), respectively, we can derive the same conditions of p and q as $p\neq4q$ and $2p\neq3q$ and then we can rewrite the exact solutions in the forms

$$u_{32}(x,t)=\mp15\sqrt{2}\Big(A_{21}(A_{22}+5\sqrt{K_1})\Big)^{\frac{1}{2}}pk^2(t)\times\left\{\frac{2(A_{22}+5\sqrt{K_1})(p^2-4pq+5q^2)+4(45q^2-p^2-8pq)A_{21}}{A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\}\pm\frac{\sqrt{6}(A_{21}(A_{22}+5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{3A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \quad (34)$$

where $z=\int_0^t k(\tau)d\tau x-\int_0^t\frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}+5\sqrt{K_1})K_{\omega1}+K_{\omega2}A_{21}}{(A_{22}+5\sqrt{K_1})K_{\omega3}+K_{\omega4}A_{21}}\right\}d\tau$ and $k(t)$ is an arbitrary function,

$$u_{42}(x,t)=\mp15\sqrt{2}\Big(A_{21}(A_{22}-5\sqrt{K_1})\Big)^{\frac{1}{2}}pk^2(t)\times\left\{\frac{2(A_{22}-5\sqrt{K_1})(p^2-4pq+5q^2)+4(45q^2-p^2-8pq)A_{21}}{3A_{21}(60(A_{22}+5\sqrt{K_1})(p+6q)-720(p+3q)A_{21})}\right\}\pm\frac{\sqrt{6}(A_{21}(A_{22}-5\sqrt{K_1}))^{\frac{1}{2}}p^2k^2(t)}{3A_{21}}\left(\frac{p-q}{p-q\exp\{-(p-q)z\}}\right)^2, \quad (35)$$

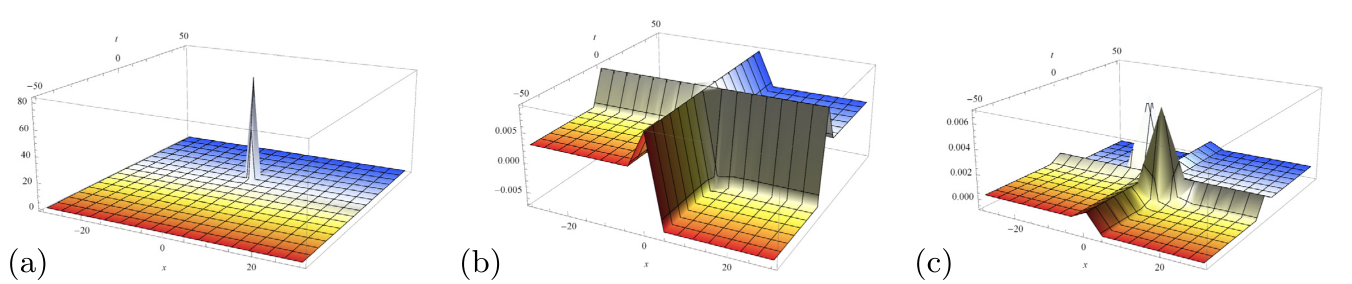

Fig. 2. Profiles of the exact solution Eq. (20): (a) one peak solution for $A_2(t)=0.1$, (b) solution with kinks for $A_2(t)=0.01$, (c) solitons-like solution for $A_2(t)=0.001$, all under $p=-1, q=-2+\sqrt{3}$.

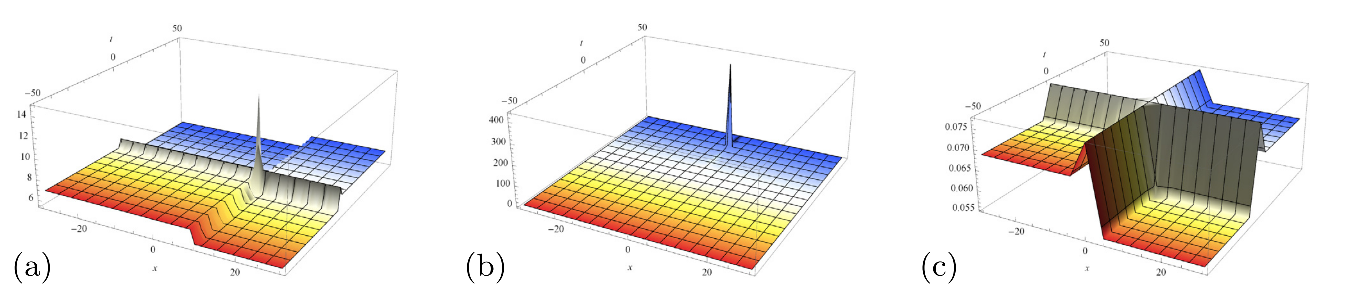

Fig. 3. Profiles of the exact solution Eq. (20): (a) soliton solution for $A_2(t)=0.1$, (b) one peak solution for $A_2(t)=0.01$, (c) solution with kinks for $A_2(t)=0.001$, all under $p=-1, q=-2-\sqrt{3}$.

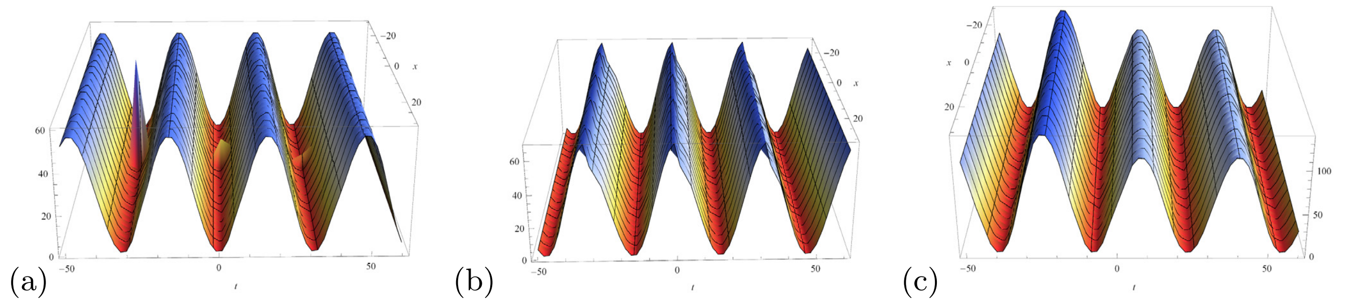

Fig. 4. Profiles of the exact solution Eq. (20): (a) periodic traveling wave solution for with peaks $A_2(t) = \sin^2(0.1t)$, (b) periodic traveling wave solution for $A_2(t) = \cos^2(0.1t)$, (c) periodic traveling wave solution for $A_2(t) = \sin^2(0.1t) + \cos^2(0.1t) = 1$, all under $p = -0.1, q = -0.1(2+\sqrt{3})$.

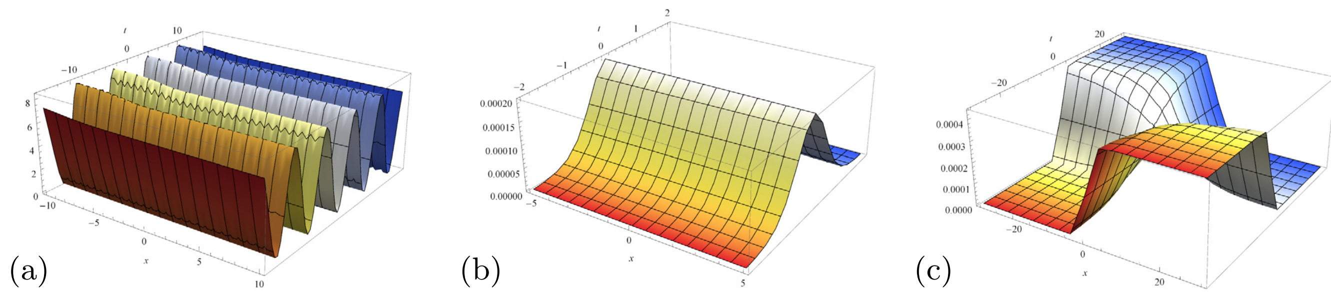

Fig. 5. Profiles of the exact solution Eq. (34): (a) periodic traveling wave solution under $p = -0.01, q = 1, k(t) = 0.5\sin(0.5t)$, (b) traveling wave solution under $p = -0.02, q = 0.01, k(t) = 1/\cosh^2(t)$, (c) traveling wave solution with kinks under $p = -0.02, q = 0.01, k(t) = 1$.

where $z = \int_0^t k(\tau)d\tau x - \int_0^t \frac{30k(\tau)}{2p-3q}\left\{\frac{(A_{22}-5\sqrt{K_1})K_{\omega 1}+K_{\omega 2}A_{21}}{(A_{22}-5\sqrt{K_1})K_{\omega 3}+K_{\omega 4}A_{21}}\right\}d\tau$ and $k(t)$ is an arbitrary function.

From the exact solution Eq. (34), we show in Fig. 5 the behaviors as the values of p, q and the spacial coefficient $k(t)$ vary: part (a) shows the periodic traveling wave motion when $p = -0.01, q = 1, k(t) = 0.5\sin(0.5t)$, part (b) shows the motion of the bell-shaped traveling wave solution with one soliton when $p = -0.02, q = 0.01, k(t) = 1/\cosh^2(t)$ and part (c) is the traveling wave solution with kinks when $p = -0.02, q = 0.01, k(t) = 1$.

Conclusion

In this paper, we have obtained new exact solutions of the fifth-order KdV type nonlinear partial differential equation using the system technique, which has been successfully applied to obtain more new general exact solutions of nonlinear partial differential equations, such as nonlinear higher-dimensional physical models. From the obtained solutions, it is noted that when we take proper variable coefficients and particular values for the physical parameters, we can demonstrate more behaviors of new exact solutions than the other existing methods.

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